A class of identities relating Whittaker and Bessel functions

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Abstract

Identities between Whittaker and modified Bessel functions are derived for particular complex orders. Certain polynomials appear in such identities, which satisfy a fourth order differential equation (not of hypergeometric type), and they themselves can be expressed as particular linear combinations of products of modified Bessel and confluent hypergeometric functions.

1 Introduction

A class of identities is derived which express Whittaker functions $W_{N,ik}(2x)$ in terms of modified Bessel functions of the second kind, where k is real, N is integer or half-integer. In this paper we concentrate on the case where $N = n + 1/2$ where n is a natural number. More explicitly, we will find that

$$W_{n+1/2,ik}(2x) = x\Lambda_n^k(x)K_{1/2+ik}(x) + x\Lambda_n^{k^*}(x)K_{1/2-ik}(x) \tag{1}$$

where $\Lambda_n^k(x)$ is a polynomial of degree n. These polynomials reduce to Laguerre polynomials when $k = 0$ and we will be able to express them as a particular linear combination of products of modified Bessel and confluent hypergeometric functions.

We should note that the $n = 0$ case of this identity was noticed in the solution of a "physical" problem; namely the energy eigenfunctions of supersymmetric quantum mechanics with an exponential potential [4,5].

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2 Proof of identities

We begin by writing down Whittaker's differential equation [1],

$$L(y) \equiv y''(x) + \left(-1 + \frac{2n+1}{x} + \frac{\frac{1}{4}+k^2}{x^2} \right) y(x) = 0, \tag{2}$$

which possesses $W_{n+1/2,ik}(2x)$ as a solution. Our strategy will be to show that $\lambda_n^k(x) K_{1/2+ik}(x) + \lambda_n^{k*}(x) K_{1/2-ik}(x)$ satisfies this differential equation for some polynomial $\lambda_n^k(x)$ and determine the polynomial as a byproduct. Then studying the asymptotics will complete the proof.

Substituting $\lambda_n^k(x) K_{1/2+ik}(x) + \lambda_n^{k*}(x) K_{1/2-ik}(x)$ into the differential equation we arrive at

$$L(\lambda_n^k K_{1/2+ik} + \text{c.c.}) = 2\lambda_n^{k'} K'_{1/2+ik} + \lambda_n^k K''_{1/2+ik} + \tag{3}$$
$$\left(\lambda_n^{k''} - \lambda_n^k + \frac{2n+1}{x}\lambda_n^k + \frac{\frac{1}{4}+k^2}{x^2}\lambda_n^k \right) K_{1/2+ik} + \text{c.c.},$$

where +c.c. means add the complex conjugate of the preceding terms. We can eliminate the second derivatives $K''_{1/2+ik}(x)$ by using Bessel's equation,

$$K''_{1/2+ik} + \frac{1}{x}K'_{1/2+ik} - \left(1 + \frac{(1/2+ik)^2}{x^2} \right) K_{1/2+ik} = 0, \tag{4}$$

and this gives,

$$\left(2\lambda_n^{k'} - \frac{\lambda_n^k}{x} \right) K'_{1/2+ik} + \left(\frac{1/2+ik}{x^2}\lambda_n^k + \frac{1+2n}{x}\lambda_n^k + \lambda_n^{k''} \right) K_{1/2+ik} + \text{c.c.}$$
$$= L(\lambda_n^k K_{1/2+ik} + \text{c.c.}). \tag{5}$$

Now, we can eliminate the first derivatives $K'_{1/2+ik}(x)$ using the identities [2]

$$xK'_\nu(x) \pm \nu K_\nu(x) = -xK_{\nu\mp1}(x), \tag{6}$$
$$K_\nu(x) = K_{-\nu}(x) \tag{7}$$

which lead to

$$\left[\lambda_n^{k''} - \frac{1+2ik}{x}\lambda_n^{k'} + \left(\frac{1+2ik}{x^2} + \frac{1+2n}{x} \right) \lambda_n^k \right] K_{1/2+ik} + \tag{8}$$

2

$$\left(\frac{\lambda_n^k}{x} - 2\lambda_n^{k'}\right) K_{1/2-ik} + \text{c.c.} = L(\lambda_n^k K_{1/2+ik} + \text{c.c.}).$$

The complex conjugate term is not independent. Since $K_{1/2+ik}^*(x) = K_{1/2-ik}(x)$ we can rewrite the whole expression above as $(...)K_{1/2+ik}+$c.c. and this can be made to vanish if the coefficient of $K_{1/2+ik}(x)$ is made to vanish; this condition corresponds to

$$\lambda_n^{k''} - \frac{1+2ik}{x}\lambda_n^{k'} + \left(\frac{1+2ik}{x^2} + \frac{1+2n}{x}\right)\lambda_n^k + \left(\frac{\lambda_n^{k*}}{x} - 2\lambda_n^{k'*}\right) = 0. \qquad (9)$$

Of course one can consider the complex conjugate version of this differential equation, and then we can view them as two linear second-order coupled differential equations for λ_n^k and λ_n^{k*}. This will imply that λ_n^k satisfies a fourth-order linear ODE. If we substitute $\lambda_n^k(x) = \sum_{m=0}^{n+1} a_m^{(n)}x^m$, we can derive a recurrence relation for the coefficients $a_m^{(n)}$. We find that

$$a_0^{(n)} = 0, \qquad (10)$$
$$m(m+1)(2m-1)(m+2ik)(m-1-2ik)a_{m+2}^{(n)} + \qquad (11)$$
$$(1+2n)m(3m^2+m-2ik)a_{m+1}^{(n)} -$$
$$4(1+2m)(n+m)(1+n-m)a_m^{(n)} = 0, \quad 1 \le m \le n-1.$$

This is a rather complicated recurrence relation,; in particular it does not generate a hypergeometric series; however given any two members of the sequence it clearly determines the rest. Thus, now we proceed to determine two of the coefficients using the asymptotics of the functions. Note that it is actually convenient to consider the recurrence relation one gets directly from the differential equation above. This is,

$$m(m-2ik)a_{m+1}^{(n)} + (1+2n)a_m^{(n)} + (1-2m)a_m^{(n)*} = 0, \quad 1 \le m \le n \quad (12)$$
$$a_{n+1}^{(n)*} - a_{n+1}^{(n)} = 0 \qquad (13)$$

and from this one gets to the second order recurrence relation above by eliminating $a_m^{(n)*}$. It is known [1] that as $x \to \infty$,

$$W_{n+1/2,ik}(2x) \sim (2x)^{n+1/2}e^{-x}, \qquad (14)$$
$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}, \qquad (15)$$

which allows us to deduce

3

$$\lambda_n^k(x) + \lambda_n^{k*}(x) \sim \frac{2^{n+1}}{\sqrt{\pi}} x^{n+1}, \tag{16}$$

and this tells us that

$$a_{n+1}^{(n)} + a_{n+1}^{(n)*} = \frac{2^{n+1}}{\sqrt{\pi}}, \tag{17}$$

providing us with enough information to solve for $a_{n+1}^{(n)}$ and we find

$$a_{n+1}^{(n)} = \frac{2^n}{\sqrt{\pi}}. \tag{18}$$

Now we turn to the asymptotics for small x. We will need [1,2]

$$W_{n+1/2,ik}(2x) \sim \frac{\Gamma(-2ik)}{\Gamma(-ik-n)}(2x)^{1/2+ik} + \text{c.c.} \qquad \text{as } x \to 0 \tag{19}$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} W_{0,\nu}(2x), \tag{20}$$

from which we can derive,

$$\lambda_n^k(x) K_{1/2+ik}(x) + \text{c.c} \sim a_1^{(n)*} 2^{ik} \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-2ik)}{\Gamma(1-ik)} x^{1/2+ik} + \text{c.c.} \tag{21}$$

and upon comparison to (19), obtain

$$a_1^{(n)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(-ik)}{\Gamma(-n-ik)} = \frac{(-1)^n}{\sqrt{\pi}}(1+ik)_n. \tag{22}$$

Thus we have derived the first and last coefficient in the polynomial, which together with the recurrence relation serve to define $\lambda_n^k(x)$ uniquely. Note that the identity is now actually proved as we have shown that $\lambda_n^k(x) K_{1/2+ik}(x) + \text{c.c}$ satisfies the same differential equation as $W_{1/2+n,ik}(2x)$ and possesses the same asymptotics and thus they must be the same function.

3 The polynomials $\Lambda_n^k(x)$

Let us consider the special case $k = 0$ for which the identity reduces to a well known one. It is clear that in this case the polynomials are real, since both $a_m^{(n)}$ and $a_m^{(n)*}$ satisfy the same recurrence relation and boundary conditions.

4

Then we see that the polynomials actually satisfy a second order differential equation and if we let $\lambda_n^0(x) = xy_n(2x)$ we find

$$zy''_n(z) + (1 - z)y'_n(z) + ny_n(z) = 0, \tag{23}$$

and thus we see that $y_n(z) = c_n L_n(z)$ where $L_n(z)$ are the Laguerre polynomials [2]. Once again, asymptotics can be used to determine the proportionality constants c_n. Therefore, since $L_n(z) \sim 1$ as $z \to 0$, and $\lambda_n^0(x) \sim (-1)^n n! x / \sqrt{\pi}$ as $x \to 0$, we see that $c_n = (-1)^n n! / \sqrt{\pi}$ and thus

$$\lambda_n^0(x) = \frac{(-1)^n n!}{\sqrt{\pi}} x\, L_n(2x). \tag{24}$$

Now, we give the fourth order equation that the polynomials satisfy. First introduce $\lambda_n^k(x) = x\Lambda_n^k(x)$; then we have

$$x\Lambda_n^{k''} + (1 - 2ik)\Lambda_n^{k'} + (1 + 2n)\Lambda_n^k - 2x\Lambda_n^{k'^*} - \Lambda_n^{k^*} = 0, \tag{25}$$

as our second order equation, and after some work one can eliminate $\Lambda_n^{k^*}$ to get the rather unsightly answer

$$a_1(x)\Lambda_n^{k''''} + a_2(x)\Lambda_n^{k'''} + a_3(x)\Lambda_n^{k''} + a_4(x)\Lambda_n^{k'} + a_5(x)\Lambda_n^k = 0, \tag{26}$$
$$a_1(x) = x^2[1 - 4ik + 4x(1 + 2n)],$$
$$a_2(x) = 4x[1 - 4ik + 3x(1 + 2n)],$$
$$a_3(x) = -16x^3(1 + 2n) + 4x^2[1 + 4ik + 8n(n + 1)] +$$
$$\qquad 4x(1 + 4k^2)(1 + 2n) + 2i(1 - 2k)(i + k)(i + 4k),$$
$$a_4(x) = -32x^2(1 + 2n) + 8x[-1 + 2n(n + 1) + 6ik] -$$
$$\qquad 4(i + k)(i + 4k)(1 + 2n),$$
$$a_5(x) = 4n(n + 1)[4x(1 + 2n) + 3(1 - 4ik)].$$

One can work out the indicial equation for this ODE (since there is a regular singular point at $x = 0$) and obtain

$$\sigma(\sigma - 1)[\sigma^2 - \sigma - 4(1 - k)(i + k)] = 0 \tag{27}$$

where the solution to the ODE behaves as x^σ as $x \to 0$. The $\sigma = 0$ solution of course corresponds to our polynomial $\Lambda_n^k(x)$. Remarkably, one can write down the general solution to this fourth order ODE, which is

$$y(x) = c_1 I_{-1/2+ik}(x)\, M_{n+1/2,ik}(2x) + c_2 I_{-1/2+ik}(x)\, W_{n+1/2,ik}(2x)$$
$$+ c_3 K_{-1/2+ik}(x)\, W_{n+1/2,ik}(2x) + c_4 K_{-1/2+ik}(x)\, M_{n+1/2,ik}(2x). \tag{28}$$

Therefore, our polynomial must be such a linear combination and one can use the asymptotics as $x \to \infty$ and $x \to 0$ to determine all the constants uniquely. After a bit of work, one finds the following:

$$c_1 = 0, \qquad c_2 = 1 + c_4 \frac{\pi \Gamma(1 + 2ik)}{\Gamma(-n + ik)}, \tag{29}$$

$$c_3 + \frac{2}{\pi} c_2 \cosh \pi k + c_4 \frac{\Gamma(-n - ik)}{\Gamma(-2ik)} = 0, \tag{30}$$

$$2c_2 + \frac{\pi c_3}{\cosh \pi k} = \frac{\Gamma(-ik)\Gamma(1/2 + ik)\Gamma(-n + ik)}{\sqrt{\pi}\Gamma(2ik)\Gamma(-n - ik)}, \tag{31}$$

which of course can be solved simultaneously. Before doing this let us examine the $k = 0$ limit. This will lead to $c_2 = 1$, $c_3 = 0$ and $c_4 = (-1)^{n+1} n!/\pi$. Using [2,3]

$$W_{n+1/2,0}(2x) = (-1)^n n! (2x)^{1/2} e^{-x} L_n^0(2x), \tag{32}$$

$$M_{n+1/2,0}(2x) = (2x)^{1/2} e^{-x} L_n^0(2x), \tag{33}$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x, \tag{34}$$

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}, \tag{35}$$

one can show that

$$\Lambda_n^0(x) = \frac{(-1)^n n!}{\sqrt{\pi}} L_n^0(2x), \tag{36}$$

as it should! Finally, solving for c_2, c_3 and c_4 we get,

$$c_2 = 1 - \frac{ik\Gamma(-ik)^2}{2^{2ik}\Gamma(2ik)\Gamma(-n - ik)^2}, \tag{37}$$

$$c_3 = -\frac{2}{\pi} \cosh \pi k + \frac{2ik\Gamma(-ik)^2 \cosh \pi k}{2^{2ik}\pi\Gamma(-n - ik)^2} + \tag{38}$$

$$\frac{\Gamma(-ik)\Gamma(-n + ik)}{\sqrt{\pi}\Gamma(2ik)\Gamma(1/2 - ik)\Gamma(-n - ik)},$$

$$c_4 = -\frac{\Gamma(-ik)^2\Gamma(-n + ik)}{2\pi 2^{2ik}\Gamma(2ik)\Gamma(-n - ik)^2} \tag{39}$$

which allows us to express the polynomial $\Lambda_n^k(x)$ in terms of modified Bessel and confluent hypergeometric functions as follows:

6

$$\Lambda_n^k(x) = c_2 I_{-1/2+ik}(x) W_{n+1/2,ik}(2x) + c_3 K_{-1/2+ik}(x) W_{n+1/2,ik}(2x)$$
$$+ c_4 K_{-1/2+ik}(x) M_{n+1/2,ik}(2x). \qquad (40)$$

Now we will give an indication as how to derive the general solution to the fourth order equation. Consider the coupled second order equation (25), and substitute the trial function $\Lambda(x) = K_{-1/2+ik}(x)F(x)$. Using Bessel's equation to eliminate second derivatives of $K_{-1/2+ik}(x)$ and the first order recurrence relation (6) to eliminate single derivatives of $K_{-1/2+ik}(x)$, we find that the trial function satisfies (25) if

$$L(F) = 0, \qquad \text{and} \qquad F^*(x) = -F(x). \qquad (41)$$

Thus possible choices for $F(x)$ are:

$$iW_{n+1/2,ik}(2x),$$
$$i(M_{n+1/2,ik}(2x) + M_{n+1/2,-ik}(2x)) \qquad \text{and}$$
$$M_{n+1/2,ik}(2x) - M_{n+1/2,-ik}(2x).$$

Hence two solutions to the fourth-order equation are $K_{-1/2+ik}(x)W_{n+1/2,ik}(2x)$ and $K_{-1/2+ik}(x)M_{n+1/2,ik}(2x)$. To get the other two we need to introduce $\tilde{I}_{-1/2+ik}(x) = I_{-1/2+ik}(x) + I_{1/2-ik}(x)$. Notice that this function satisfies the same Bessel equation as $K_{-1/2+ik}(x)$ does, and also a very similar first order recurrence relation, namely $x\tilde{I}'_{-1/2+ik}(x) - (-1/2+ik)\tilde{I}_{-1/2+ik}(x) = x\tilde{I}^*_{-1/2+ik}(x)$ (this equation has a minus on the RHS for $K_{-1/2+ik}(x)$, see (6)). Substituting the trial function $\tilde{I}_{-1/2+ik}(x)G(x)$ leads to a similar condition on $G(x)$ as we obtained for $F(x)$:

$$L(G) = 0, \qquad \text{and} \qquad G^*(x) = G(x). \qquad (42)$$

Thus $\tilde{I}_{-1/2+ik}(x)W_{n+1/2,ik}(2x)$ and $\tilde{I}_{-1/2+ik}(x)M_{n+1/2,ik}(2x)$ solve the fourth-order equation. Using the other two solutions, this means $I_{-1/2+ik}(x)W_{n+1/2,ik}(2x)$ and $I_{-1/2+ik}(x)M_{n+1/2,ik}(2x)$ are also solutions to the fourth order equation, and hence we have completed the proof.

4 Related identities

We have derived an identity for $N = n+1/2$. There also exist similar identities for $N = -n - 1/2$. The case $N = \pm n$ is also of interest; in this case the identities look like $W_{n,ik}(2x) = \sqrt{x}p_n(x)K_{1+ik}(x) + \sqrt{x}q_n(x)K_{ik}(x)$, where $p_n(x)$ and $q_n(x)$ are polynomials of degree n, but we shall not go through the details here. Also note that similar identities probably hold between $M_{N,ik}(2x)$

and $\tilde{I}_{-1/2+ik}(x)$, since the function $\tilde{I}_\nu(x)$, like $K_\nu(x)$, is symmetric in the order and hence $\tilde{I}^*_{-1/2+ik}(x) = \tilde{I}_{1+(-1/2+ik)}(x)$, just like for $K_{-1/2+ik}(x)$, which was an important property we used in the proof.

Acknowledgments

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